

# ‘Superluminal’ Photon Propagation in QED in Curved Spacetime is Dispersive and Causal

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**ABSTRACT:** It is now well-known that vacuum polarisation in QED can lead to superluminal low-frequency phase velocities for photons propagating in curved spacetimes. In a series of papers, we have shown that this quantum phenomenon is dispersive and have calculated the full frequency dependence of the refractive index, explaining in detail how causality is preserved and various familiar results from quantum field theory such as the Kramers-Kronig dispersion relation and the optical theorem are realised in curved spacetime. These results have been criticised in a recent paper by Akhouri and Dolgov [1], who assert that photon propagation is neither dispersive nor necessarily causal. In this note, we point out a series of errors in their work which have led to this false conclusion.

1. One of the important simplifications in our analysis of photon propagation in curved spacetime is the insight that, in the eikonal and ‘weak curvature’ approximations (see section 2), the background may be replaced by its Penrose plane wave limit around the null geodesic describing the classical trajectory. The first main claim of [1] is that such plane waves are too simple to manifest the vacuum polarisation induced modifications to photon propagation discovered by Drummond and Hathrell [2]. This claim is simply not true: in fact plane waves have precisely the data that is encoded in the Drummond-Hathrell result.

The origin of superluminal low-frequency propagation is the effective action for QED in curved spacetime [2]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{\pi} \frac{1}{m^2} \left( a_1 R F_{\mu\nu} F^{\mu\nu} + a_2 R_{\mu\nu} F^{\mu\lambda} F^{\nu}_{\lambda} + a_3 R_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \right) + \dots \quad (1)$$

The dots indicate that this effective action is the first term in a derivative expansion, so results deduced from it are valid only for low-frequency propagation. The refractive index derived from (1) for a photon with wave-vector  $k^\mu = \omega \hat{k}^\mu$ , where  $\omega$  is the frequency, is

$$n_{ij} = \delta_{ij} + \frac{\alpha}{\pi} \frac{1}{m^2} \left( c_1 \delta_{ij} R_{\mu\nu} \hat{k}^\mu \hat{k}^\nu + c_2 R_{\mu i \nu j} \hat{k}^\mu \hat{k}^\nu \right), \quad (2)$$

where the indices  $i, j$  label the two spacelike polarisation directions. The constant coefficients  $c_1, c_2$  are simply related to the known coefficients  $a_2, a_3$  in the effective action. If we now introduce a null coordinate  $u$  along the direction  $k^\mu$ , then

$$n_{ij} = \delta_{ij} + \frac{\alpha}{\pi} \frac{1}{m^2} \left( c_1 \delta_{ij} R_{uu} + c_2 R_{uiuj} \right). \quad (3)$$

So the Drummond-Hathrell result depends on the curvature components  $R_{uiuj}$  and  $R_{uu}$ .

Now, a plane wave metric in Brinkmann coordinates takes the form

$$ds^2 = 2du dv + h_{ij}(u) y^i y^j du^2 + dy^i dy^i. \quad (4)$$

where  $(u, v)$  are null coordinates and  $y^i$  are transverse spacelike coordinates. The non-vanishing components of the Riemann tensor describe the wave profile,  $R_{uiuj} = h_{ij}$ . So if we apply the Drummond-Hathrell formula (2) to a wave propagating along the geodesic  $u$  with  $v = y^i = 0$ , that is  $k^u = \omega$ ,  $k^v = 0$  and  $k^i = 0$ , we find a non-vanishing effect involving precisely the curvature components  $R_{uiuj}$  that are non-trivial in a plane wave spacetime.

Of course, this is apparent in our work [3–6], since we have demonstrated that the low-frequency limit of our full, dispersive refractive index formula reproduces

the original Drummond-Hathrell result. The error in [1] is in their eq.(12), which is derived simply from the equation of motion in a Ricci flat plane wave background. This should read:

$$2\partial_u\partial_v A_k + (g^{vv} - 8\hat{a}_3 R_{ukuk})\partial_v^2 A_k = 0, \quad (5)$$

with  $\hat{a}_3 = \frac{\alpha}{\pi} \frac{1}{m^2} a_3$ , leading (since of course  $g^{vv} = g_{uu} = 0$ ) to the usual result (3), and *not*, as quoted in [1],

$$(2 + 4\hat{a}_3 R_{ukuk})\partial_u\partial_v A_k + g^{vv}\partial_v^2 A_k = 0, \quad (6)$$

The error is just a simple mistake of raising/lowering indices with the off-diagonal metric in light cone coordinates  $u, v$ . However, it leads the authors of [1] to the important but manifestly wrong conclusion that the Penrose limit fails to capture the Drummond-Hathrell effect.

The intuitive reason why the Penrose limit captures the essential geometry is most evident in the worldline, or proper-time, representations of the propagator where it is clear that the vacuum polarisation depends on the geometry of geodesic fluctuations around the original photon trajectory [4, 5]. This is precisely what is captured by the Penrose limit and is encoded in the Van Vleck-Morette matrix, which plays a key role in our analysis.

More recently [7], we have also made an independent check on the validity of the Penrose limit. In the case of a background spacetime  $dS_3 \times \mathbf{R}$ , we can evaluate the refractive index, or spectral density, exactly. Then, taking the appropriate WKB limits (section 2), we recover precisely the result we obtain by using the Penrose limit directly, in this case a conformally-flat symmetric plane wave with  $R_{uu} < 0$ . This gives a highly non-trivial confirmation that the Penrose limit does indeed retain the essential geometry of the full background field relevant for determining the vacuum polarisation corrections to photon propagation.

**2.** Photon propagation with vacuum polarisation in curved spacetime is characterised by three length scales and it is important to be clear about the regimes of curvature and frequency where our results are valid. Let  $1/\sqrt{\mathcal{R}}$  denote the generic curvature length scale (so  $\mathcal{R}$  can represent a curvature or derivatives of curvature),  $\omega$  the photon frequency and  $m$  the electron mass. We work in the ‘weak curvature’ limit  $\sqrt{\mathcal{R}} \ll m$  (so the curvature scale is much greater than the electron Compton wavelength) and in the usual geometric optics, or ‘eikonal’ approximation  $\omega \gg \sqrt{\mathcal{R}}$  (so the photon wavelength is much less than the curvature scale). This defines the WKB limit in which the Penrose limit is a good approximation to the general background spacetime.

An important parameter in our analysis is the remaining dimensionless variable  $\omega\sqrt{\mathcal{R}}/m^2$  and we need to consider all values of this to calculate the full frequency dependence of the refractive index. Causality is related to the limit  $n(\infty)$  since this is identified with the wavefront velocity of the light wave. Since ref. [1] misrepresents the role of this parameter in our analysis of the high frequency behaviour of  $n(\omega)$ , and therefore wrongly criticises the implications of our results for causality, we spell out the parameter dependence of our results here in explicit detail.

Our result for the refractive index, taking scalar QED for simplicity, is (see, e.g. eq.(4.42) of [5])

$$n(u; \omega) = 1 + \frac{\alpha}{2\pi\omega} \int_0^1 d\xi \xi(1-\xi) \mathcal{F}\left(u; \frac{m^2}{2\omega\xi(1-\xi)}\right), \quad (7)$$

with

$$\mathcal{F}(u; z) = \int_0^{\infty-i\epsilon} \frac{dt}{t^2} i e^{-izt} \left[ 1 - \Delta(u, u-t) \sqrt{\det \Delta(u, u-t)} \right], \quad (8)$$

The curvature dependence is embedded in the Van Vleck-Morette matrix  $\Delta(u, u-t)$  which is a function of  $\sqrt{\mathcal{R}}t$ . For small  $t$ , it can be expanded in schematic form as

$$\Delta(u, u-t) \rightarrow 1 + \sum_{n=1}^{\infty} \mathcal{R}^n(u) t^{2n}, \quad (9)$$

where to linear order, the relevant curvature component is  $R_{uiuj}$ . (The vector notation takes account of the polarisation dependence,  $i, j = 1, 2$ , and will be dropped from now on for simplicity.) A simple rescaling  $t \rightarrow t/z$ , or alternatively  $t \rightarrow t/\sqrt{\mathcal{R}}$ , shows immediately that eq.(7) can be written in the (clearly equivalent) forms:

$$n = 1 + \frac{\alpha}{\pi} \frac{m^2}{\omega^2} F\left(\frac{\omega\sqrt{\mathcal{R}}}{m^2}\right) = 1 + \frac{\alpha}{\pi} \frac{\sqrt{\mathcal{R}}}{\omega} G\left(\frac{m^2}{\omega\sqrt{\mathcal{R}}}\right), \quad (10)$$

where the function  $F$  (or  $G$ ) is calculated non-perturbatively. This allows us to access both the low and high frequency limits of the refractive index.

At low frequencies, we can expand  $F$  as a power series in small  $\frac{\omega\sqrt{\mathcal{R}}}{m^2}$  using the VVM expansion (9). This starts with a term of  $\mathcal{O}\left(\frac{\omega\sqrt{\mathcal{R}}}{m^2}\right)^2$ , giving the refractive index in the schematic form

$$n = 1 + \frac{\alpha}{\pi} \frac{\mathcal{R}}{m^2} \left( 1 + \mathcal{O}\left(\frac{\omega\sqrt{\mathcal{R}}}{m^2}\right) \right). \quad (11)$$

The first term, independent of frequency, reproduces the Drummond-Hathrell result. Of particular interest is the next term in the expansion which, if present, gives

a contribution to the imaginary part of the refractive index  $\text{Im } n(\omega)$  of the form  $\omega \partial_u R / m^4$ . These contributions can be reproduced by an extension of the DH effective action, discussed below.

At high frequencies, we can use the second form of (10) and expand  $G$  for small  $\frac{m^2}{\omega \sqrt{\mathcal{R}}}$ . Here, the first term is found to be of  $\mathcal{O}(1)$  and the expansion may contain logarithms as well as powers. We therefore find

$$n = 1 + \frac{\alpha \sqrt{\mathcal{R}}}{\pi \omega} + \dots, \quad (12)$$

with  $n(\infty) = 1$  as expected in a causal theory.

All this is illustrated quite explicitly in a simple example discussed in [5], the conformally flat symmetric plane wave, where we have an exact analytic expression for  $\mathcal{F}(z)$  in (8). In this model, the wave profile function is  $h_{ij} = \sigma^2 \delta_{ij}$ , so  $\sqrt{\mathcal{R}} = \sigma$ . We find:

$$\mathcal{F}(z) = \sigma \left( \frac{z}{\sigma} \psi \left( 1 + \frac{z}{2\sigma} \right) - \frac{z}{\sigma} \log \frac{z}{2\sigma} - 1 \right). \quad (13)$$

For large  $z \sim \frac{m^2}{\omega}$ , i.e. low frequency, the digamma function has the expansion

$$\psi \left( 1 + \frac{z}{2\sigma} \right) = \log \frac{z}{2\sigma} + \frac{\sigma}{z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n} \left( \frac{2\sigma}{z} \right)^{2n}, \quad (14)$$

where  $B_{2n}$  are the Bernoulli numbers, and so

$$\mathcal{F}(z) = -\frac{1}{3} \frac{\sigma^2}{z} \left( 1 + \mathcal{O} \left( \frac{\sigma}{z} \right)^2 \right). \quad (15)$$

Substituting in (7) gives the refractive index [5]

$$n(\omega) = 1 - \frac{\alpha}{90\pi} \frac{\sigma^2}{m^2} \left( 1 + \mathcal{O} \left( \frac{\omega^2 \sigma^2}{m^4} \right) \right). \quad (16)$$

Notice that in this case the expansion has no term linear in  $\omega$  and indeed  $\text{Im } n(\omega) = 0$  in this model.

In the opposite limit of small  $z$ , i.e. high frequency, the relevant expansion of the digamma function is

$$\psi \left( 1 + \frac{z}{2\sigma} \right) = \sum_{n=0}^{\infty} \frac{\psi_n(1)}{\Gamma(n+1)} \left( \frac{z}{2\sigma} \right)^n, \quad (17)$$

giving

$$\mathcal{F}(z) = \sigma \left( -1 - \frac{z}{\sigma} \left( \log \frac{z}{2\sigma} + \gamma \right) + \mathcal{O} \left( \frac{z}{\sigma} \right)^2 \right). \quad (18)$$

The refractive index is therefore [5]

$$n(\omega) = 1 - \frac{\alpha}{12\pi} \frac{\sigma}{\omega} \left( 1 + \mathcal{O}\left(\frac{m^2}{\omega\sigma}\right) \right), \quad (19)$$

where the first correction also includes the logarithm from (18). This example demonstrates precisely how both the low and high frequency limits of the refractive index are realised in terms of the basic parameters  $\sqrt{\mathcal{R}}$ ,  $m$  and  $\omega$  and gives the full non-perturbative function of  $\omega\sqrt{\mathcal{R}}/m^2$  that interpolates between them. It explicitly refutes the claim of ref. [1] that the corrections to the low-frequency value  $n(0)$  vanish in the high-frequency limit leaving  $n(\infty) \neq 1$  with the associated problems with causality.

**3.** In flat spacetime, an important role is played by the Kramers-Kronig dispersion relation

$$n(\infty) = n(0) - \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega} \text{Im } n(\omega) \quad (\text{flat spacetime}). \quad (20)$$

In this case,  $\text{Im } n(\omega) > 0$  by virtue of the optical theorem, which relates it to the forward scattering cross-section, so  $n(\infty) < n(0)$  or equivalently  $v_{\text{ph}}(\infty) > v_{\text{ph}}(0)$ . A superluminal low-frequency phase velocity in flat spacetime would therefore imply the wavefront velocity  $v_{\text{ph}}(\infty)$  exceeds  $c$ , violating causality.<sup>1</sup>

However, in curved spacetime neither the Kramers-Kronig dispersion relation nor the optical theorem remain true in their normal flat spacetime forms, and the usual insights based on (20) are simply wrong. Nevertheless, ref. [1] continues to use (20) with the inevitable mistaken conclusions. Moreover, a confusion is made (eq.(2) of ref. [1]) between purely geometric and vacuum polarisation effects on  $\text{Im } n(\omega)$ .

To explain this second point, note that the eikonal approximation for the solution to the  $\mathcal{O}(\alpha)$  corrected wave equation is (e.g. [6])

$$A_\mu(x) = \mathcal{A}(x) \epsilon_\mu(x) e^{-i\omega \left( V - \int^u du (n(u;\omega) - 1) \right)}, \quad (21)$$

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<sup>1</sup>An interesting, but quite separate, question is whether causality would be violated in a model in which the strong equivalence principle is violated explicitly by tree-level curvature couplings and we can use a Drummond-Hathrell type action directly to analyse signal propagation and causality.

In ref. [8], it was pointed out that this is related to ‘*stable causality*’ and depends on the global nature of the background spacetime. The question is whether the spacetime still admits a global Killing vector which is timelike with respect to the extended light cones of the DH action. Ref. [1] disputes this, repeating the proposal of Dolgov and Novikov [9] that the superluminal DH effect permits the construction of a ‘time machine’ which can be realised, for example, by two black holes in relative motion where each exhibits superluminal propagation in its vicinity. However, it has already been explained in [10], section 5, (see also [11]) that this proposed time machine does *not* work, precisely because of the fact that the DH effect is dependent on the local spacetime curvature (a fact dismissed in [1] as a ‘trivial complication’), which invalidates the argument of [9].

where  $V$  is a Rosen coordinate. The amplitude  $\mathcal{A}(x)$  satisfies the classical equation  $\partial_u \log \mathcal{A} = -\hat{\theta}$  along the null geodesic, where  $\hat{\theta}$  is the optical scalar in the Raychoudhuri equations [6] which describes the expansion or contraction of the null geodesic congruence describing the wave propagation. The amplitude can therefore increase or decrease due to this classical geometric effect, which should be clearly distinguished from the  $\mathcal{O}(\alpha)$  change to the amplitude which would be induced by an imaginary part of  $n(u; \omega)$  in (21). This would correspond to genuine dispersion due to particle creation if  $\text{Im } n(\omega) > 0$  or the more subtle photon ‘undressing’ effect described carefully in ref. [7] if  $\text{Im } n(\omega) < 0$ . It is only this latter effect that is related to vacuum polarisation, the Kramers-Kronig dispersion relation and the optical theorem. The assertion in [1] that such effects are ‘clearly negligible’ in this context completely misses the point, since the whole phenomenon of vacuum polarisation induced corrections to photon propagation takes place at this order.

The form of the Kramers-Kronig dispersion relation which remains valid in curved spacetime is [5]

$$n(u; \infty) = n(u; 0) - \frac{1}{i\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} n(u, \omega) , \quad (22)$$

This assumes only that  $n(u; \omega)$  is bounded at infinity and analytic in the upper half  $\omega$ -plane, as required by causality. In order to recover the conventional form (20), we need to assume also that we have translation invariance along the geodesic, so that  $n(\omega)$  is an even function of  $\omega$ , and that it satisfies *real analyticity*,  $n(\omega^*) = n(\omega)^*$ . This is described in full detail in [5]. One of the key discoveries of our work is that this second property does not hold in general, due to a novel analytic structure of the refractive index, or vacuum polarisation, in curved spacetime. This is due to additional singularities and branch cuts (evident in (13)) related to the existence of conjugate points on the null geodesics, i.e. points which may be joined by a continuous family of geodesics infinitesimally close to the original one. (In the world-line formalism, these correspond to zero-modes in the fluctuations around the classical null geodesic path.)

The novel analytic structure we have uncovered in curved spacetime may also give rise to a non-perturbative imaginary part for the refractive index, distinct from the  $\omega \partial_u R / m^4$  type terms described above. These contributions to  $\text{Im } n(\omega)$  can arise even in spacetimes where the curvature is translation invariant along the photon’s null geodesic, as e.g. the Ricci-flat symmetric plane wave [5].

4. These entirely non-perturbative effects cannot be seen in a local effective action of the Drummond-Hathrell type, even when it is extended to include higher derivatives to all orders [6, 12, 13]. However, this is sufficient to capture the perturbative imaginary parts proportional to  $\omega \partial_u R / m^4$ . Since this is also relevant for the central argument of ref. [1] (see section 5), we recall some details of the derivation here. The generalised effective action is:

$$\begin{aligned} \Gamma = & \int d^4x \sqrt{-g} \left[ -\frac{1}{4} Z F_{\mu\nu} F^{\mu\nu} + D_\mu F^{\mu\lambda} \vec{d}_0 D_\nu F^\nu{}_\lambda \right. \\ & + \frac{1}{m^2} \left( \vec{a}_0 R F_{\mu\nu} F^{\mu\nu} + \vec{b}_0 R_{\mu\nu} F^{\mu\lambda} F^\nu{}_\lambda + \vec{c}_0 R_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \right) \\ & + \frac{1}{m^4} \left( \vec{a}_1 R D_\mu F^{\mu\lambda} D_\nu F^\nu{}_\lambda + \vec{b}_1 R_{\mu\nu} D_\lambda F^{\lambda\mu} D_\rho F^{\rho\nu} \right. \\ & \left. \left. + \vec{b}_2 R_{\mu\nu} D^\mu F^{\lambda\rho} D^\nu F_{\lambda\rho} + \vec{b}_3 R_{\mu\nu} D^\mu D^\lambda F_{\lambda\rho} F^{\rho\nu} + \vec{c}_1 R_{\mu\nu\lambda\rho} D_\sigma F^{\sigma\rho} D^\lambda F^{\mu\nu} \right) \right] \end{aligned} \quad (23)$$

In this formula, the  $\vec{a}_n$ ,  $\vec{b}_n$ ,  $\vec{c}_n$  are known ‘form factor’ functions of three operators, *i.e.*

$$\vec{a}_n \equiv a_n \left( \frac{D_{(1)}^2}{m^2}, \frac{D_{(2)}^2}{m^2}, \frac{D_{(3)}^2}{m^2} \right), \quad (24)$$

where the first entry  $D_{(1)}^2$  acts on the first following term (the curvature), etc.

The refractive index derived from (23) is [13]

$$n_{ij}(\omega) = \delta_{ij} + \delta_{ij} \frac{\alpha}{\pi} \frac{1}{m^2} c_1 \left( \frac{2i\omega \hat{k} \cdot D}{m^2} \right) R_{uu} + \frac{\alpha}{\pi} \frac{1}{m^2} c_2 \left( \frac{2i\omega \hat{k} \cdot D}{m^2} \right) R_{uiuj}, \quad (25)$$

where the constant coefficients  $c_1$ ,  $c_2$  of (3) are replaced by functions of the operator  $\hat{k} \cdot D \sim \partial_u$ , which describes the variation of the curvature tensors along the classical null geodesic. The linear terms in the expansion of  $c_1$  and  $c_2$  give the contributions to  $\text{Im } n(\omega)$  of  $\mathcal{O}(\frac{\omega \sqrt{R}}{m^2} \frac{R}{m^2})$  discussed following (11). We have checked that these agree with those derived using our VVM-based method [6].

To understand the origin of this expression, consider as an example the term

$$\frac{1}{m^4} \int d^4x \sqrt{-g} R_{\mu\nu\lambda\rho} F^{\mu\nu} D^2 F^{\lambda\rho} \quad (26)$$

in the effective action, incorporating a single power of  $D^2$  from the form factor. The corresponding contribution to the equation of motion for  $D_\mu F^{\mu\nu}$  is

$$\frac{1}{m^4} \left[ -2(D^2 R_\mu{}^\nu{}_{\lambda\rho}) D^\mu F^{\lambda\rho} - 4(D_\sigma R_\mu{}^\nu{}_{\lambda\rho}) D^\sigma D^\mu F^{\lambda\rho} - 4R_\mu{}^\nu{}_{\lambda\rho} D^2 D^\mu F^{\lambda\rho} \right]. \quad (27)$$



The key point is that only the second term survives at leading order in the eikonal approximation, where it contributes to the  $c_2$  term in (25). The first term is suppressed by a power of  $\mathcal{R}/m^2$  while the third is of  $\mathcal{O}(k^2)$ , which is suppressed by a power of  $\alpha$  since the photon is on-shell before the radiative corrections. By contrast, the second term is of only of relative order  $\mathcal{O}(\frac{\omega\sqrt{\mathcal{R}}}{m^2})$ , the now familiar parameter.

**5.** The main constructive part of ref. [1] is based on a 1994 paper by Khriplovich [14] which argues on the basis of a proposed general form for the graviton-photon-photon vertex that all corrections to the refractive index beyond the low-frequency contribution  $n(0)$  vanish. Although at first sight plausible, it is apparent that this argument misses the contributions of the form  $k^\mu D_\mu R/m^4 \sim \omega \partial_u R/m^4$  that we have found in both the effective action and VVM approaches. This was pointed out already in ref. [13]. Of course, Khriplovich's method could not in any case detect the non-perturbative contributions following from the occurrence of conjugate points on the photon's null geodesic.

Although it is obvious from our explicit construction of  $n(\omega)$  in many examples that Khriplovich's argument cannot be correct, the loophole in the analysis of [14] is in fact quite subtle. Following [14], we define the graviton-photon-photon vertex in momentum space on a flat background:

$$(2\pi)^4 \delta^{(4)}(q+k+k') \Gamma(q, k, k') = \tilde{h}^{\mu\nu}(q) \int dz e^{iz \cdot q} \langle \epsilon(k) | T_{\mu\nu}(z) | \epsilon(k') \rangle, \quad (28)$$

where  $\epsilon(k)$  is the photon polarisation and  $\tilde{h}_{\mu\nu}(q)$  is the external graviton field. Then, according to [14] (see also [2, 15]), the most general form of the vertex to  $\mathcal{O}(1/m^2)$  can be written as

$$\begin{aligned} \Gamma(q, k, k') = & a_1(q^2, k^2, k'^2) \tilde{R}^h(q) \tilde{F}_{\mu\nu}(k) \tilde{F}^{\mu\nu}(k') + a_2(q^2, k^2, k'^2) \tilde{R}_{\mu\nu}^h(q) \tilde{F}^{\mu\lambda}(k) \tilde{F}^\nu{}_\lambda(k') \\ & + a_3(q^2, k^2, k'^2) \tilde{R}_{\mu\nu\lambda\rho}^h(q) \tilde{F}^{\mu\nu}(k) \tilde{F}^{\lambda\rho}(k'), \end{aligned} \quad (29)$$

where  $\tilde{F}_{\mu\nu}(k) = k_\mu \epsilon_\nu(k) - k_\nu \epsilon_\mu(k)$  and  $\tilde{R}^h(q)$  is the Fourier transform of the Ricci scalar evaluated with the metric identified with the graviton field, etc. The  $a_i(q^2, k^2, k'^2)$  are form factors. The similarity with the effective action (23) is clear. Higher-order Lorentz structures of  $\mathcal{O}(1/m^4)$  with four momenta can be read off from (23) (see also, e.g. [16]).

The argument of [1, 14] now proceeds by claiming that the imaginary part of  $\Gamma$ , with the photons taken on-shell, only has a contribution from  $q^2 = 0$  so that  $\text{Im}\Gamma(q^2, 0, 0) \sim \delta(q^2)$ . Since the vertex has no dependence on the photon frequencies, it follows that the refractive index itself must be independent of frequency and that  $n(\omega) = n(0)$  for all  $\omega$ , i.e. the Drummond-Hathrell effect is non-dispersive.

To see what is wrong with applying this argument in curved spacetime, notice that for a slowly varying background gravitational field, the Fourier transform  $\tilde{R}(q)$  of the curvature is in fact very singular at  $q = 0$ . If we expand (dropping indices for clarity):

$$R(z) = R(0) + z^\mu \partial_\mu R(0) + \dots, \quad (30)$$

then

$$\tilde{R}(q) = R(0)\delta^{(4)}(q) - i\partial_\mu R(0)\frac{\partial}{\partial q_\mu}\delta^{(4)}(q) + \dots. \quad (31)$$

To see the effect of this singular behaviour involving derivatives of the delta function, substitute (31) and integrate a typical vertex term over  $q$ :

$$\begin{aligned} & \int d^4q \, a(q^2, k^2, (k+q)^2) \tilde{R}(q) \tilde{F}(k) \tilde{F}(-k-q) \\ &= a(0, k^2, k^2) R(0) \tilde{F}(k) \tilde{F}(-k) - a^{(3)}(0, k^2, k^2) 2ik^\mu \partial_\mu R(0) \tilde{F}(k) \tilde{F}(-k) \\ &+ a(0, k^2, k^2) i\partial_\mu R(0) \tilde{F}(k) \frac{\partial}{\partial k_\mu} \tilde{F}(-k), \end{aligned} \quad (32)$$

where  $a^{(3)}$  denotes differentiation w.r.t. the third argument. Of these terms, the first is obviously just the zero-momentum Drummond-Hathrell contribution, while the third is lower order in the eikonal approximation. The second term, however, can contribute at leading order even on-shell and gives rise to an imaginary part for the refractive index proportional to  $k^\mu \partial_\mu R(0)$ , i.e. precisely the frequency-dependent contribution of  $\omega \partial_u R/m^4$  type that we have already identified. It arises in essentially the same way as illustrated for the effective action in (26),(27).

What this shows is that when the Khriplovich argument is applied to curved spacetime backgrounds, it is necessary to keep careful track of the full singularity structure, including derivatives of the momentum-space delta function associated with the graviton insertion. Yet again, we find that a careful analysis shows that the refractive index for photon propagation on curved spacetime *is* dispersive, with a low-frequency expansion (11) which may have an imaginary part proportional to  $k^\mu D_\mu R/m^4$ .

**6.** Finally, the question of whether QED in curved spacetime is causal is ultimately determined by whether the commutator, or Pauli-Jordan, Green function  $iG_{\mu\nu}(x, x') = \langle 0|[A_\mu(x), A_\nu(x')]|0\rangle$  vanishes outside the light cone. This was demonstrated in ref. [5] by an explicit construction, contradicting the claims in [1]. In fact, we were able to show that the commutator in scalar QED in the Penrose plane wave

limit, incorporating vacuum polarisation, can be written as

$$G_{ij}(x, x') = 2\frac{\alpha}{\pi} \int_{u'}^u d\tilde{u} \int_0^{u-u'} \frac{dt}{t^2} \int_0^1 d\xi \xi(1-\xi) \times \Delta_{ij}(\tilde{u}, \tilde{u}-t) \sqrt{\det \Delta(\tilde{u}, \tilde{u}-t)} G\left(\frac{m^2 t}{2\xi(1-\xi)(u-u')}; x, x'\right) + \dots, \quad (33)$$

where the omitted terms are independent of the curvature. Here,  $G(m^2; x, x')$  is the commutator Green function for a free massive scalar field, which clearly has support only on or inside the light cone. Since  $G_{ij}(x, x')$  therefore vanishes outside the light cone, causality is manifest.

It is interesting to relate this to the refractive index. If we take  $x = (u, V, 0, 0)$  and  $x' = (u' \rightarrow -\infty, 0, 0, 0)$  to be two points on the classical photon trajectory, we find (note that since we are taking  $u > u'$ , the commutator and retarded Green functions coincide):

$$G_{ij}(x; x') \sim \int_{-\infty}^u d\tilde{u} \int_{-\infty}^{\infty} d\omega n_{ij}(\tilde{u}; \omega) e^{-i\omega V}. \quad (34)$$

The properties we have established for the refractive index, in particular analyticity in the upper half  $\omega$ -plane, now show that the commutator (retarded) Green function vanishes when  $V < 0$ , i.e. outside the light cone.

In summary, despite the many erroneous claims to the contrary in ref. [1], we have clearly established [3–6] that photon propagation in QED in curved spacetime is indeed dispersive and causal.

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